

POINTWISE ESTIMATES OF BREZIS–KAMIN TYPE FOR SOLUTIONS OF SUBLINEAR ELLIPTIC EQUATIONS

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ABSTRACT. We study quasilinear elliptic equations of the type

$$-\Delta_p u = \sigma u^q \quad \text{in } \mathbb{R}^n,$$

where $\Delta_p u = \nabla \cdot (\nabla u |\nabla u|^{p-2})$ is the p -Laplacian (or a more general \mathcal{A} -Laplace operator $\operatorname{div} \mathcal{A}(x, \nabla u)$), $0 < q < p-1$, and $\sigma \geq 0$ is an arbitrary locally integrable function or measure on \mathbb{R}^n .

We obtain necessary and sufficient conditions for the existence of positive solutions (not necessarily bounded) which satisfy global pointwise estimates of Brezis–Kamin type given in terms of Wolff potentials. Similar problems with the fractional Laplacian $(-\Delta)^\alpha$ for $0 < \alpha < \frac{n}{2}$ are treated as well, including explicit estimates for radially symmetric σ . Our results are new even in the classical case $p = 2$ and $\alpha = 1$.

1. INTRODUCTION

We study quasilinear equations of the type

$$(1.1) \quad \begin{cases} -\Delta_p u = \sigma u^q & \text{in } \mathbb{R}^n, \\ \liminf_{x \rightarrow \infty} u(x) = r, & u > 0, \end{cases}$$

where $\Delta_p u = \nabla \cdot (\nabla u |\nabla u|^{p-2})$ is the p -Laplacian, $0 < q < p-1$, $r \geq 0$, and $\sigma \in M^+(\mathbb{R}^n)$; here $M^+(\mathbb{R}^n)$ denotes the class of all nonnegative locally finite Borel measures on \mathbb{R}^n . Our main results are new, in particular, for nonnegative $\sigma \in L^1_{\text{loc}}(\mathbb{R}^n)$.

We prove the existence of distributional solutions to (1.1) and obtain sharp global pointwise bounds of solutions of Brezis–Kamin type (see [BK92] for bounded solutions in the case $p = 2$) in terms of Wolff potentials. Necessary and sufficient conditions on σ for the existence of solutions to (1.1) which satisfy such estimates will be presented.

We also obtain similar results for the fractional Laplace equation,

$$(1.2) \quad \begin{cases} (-\Delta)^\alpha u = \sigma u^q & \text{in } \mathbb{R}^n, \\ \liminf_{x \rightarrow \infty} u(x) = r, & u > 0, \end{cases}$$

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where $0 < q < 1$ and $0 < \alpha < \frac{n}{2}$. In particular, for radially symmetric σ , explicit conditions which ensure the existence of radial solutions together with sharp bilateral bounds of solutions are given.

In the classical case $p = 2$, the sublinear problem (1.1), or equivalently (1.2) with $\alpha = 1$, was studied in [BK92], where a necessary and sufficient condition for the existence of *bounded* solutions was given, together with uniqueness. Of specific interest to us are the following global pointwise estimates of bounded solutions obtained in [BK92]:

$$(1.3) \quad c^{-1} (\mathbf{I}_2 \sigma)^{\frac{1}{1-q}} \leq u \leq c \mathbf{I}_2 \sigma,$$

where c is a positive constant, and $\mathbf{I}_2 \sigma = (-\Delta)^{-1} \sigma$ is the Newtonian potential of σ .

The fractional Laplace equation (1.2) was studied recently by Punzo and Terrone [PT14]. They considered bounded solutions and obtained similar pointwise estimates in terms of Riesz potentials $\mathbf{I}_{2\alpha} \sigma = (-\Delta)^{-\alpha} \sigma$ under the restriction $0 < \alpha < \min\{1, \frac{n}{4}\}$.

Recently, we characterized the existence of finite energy solutions [CV14a] and arbitrary distributional solutions [CV14b], and gave matching upper and lower pointwise estimates of solutions using new intrinsic potentials of Wolff type, both for equation (1.1) with $0 < q < p - 1$, and (1.2) with $0 < q < 1$ and $0 < \alpha < \frac{n}{2}$. However, the definition and applications of intrinsic Wolff potentials are complicated.

The main goal of this paper is to prove the existence of a broad class of solutions, not necessarily bounded, and obtain Brezis–Kamin type estimates similar to (1.3) in terms of the usual Wolff potentials defined in [HW83] (see also [AH96], [Maz11]), or classical Riesz potentials in the case $p = 2$ or $\alpha = 1$, both for equation (1.1) with $0 < q < p - 1$, and (1.2) with $0 < q < 1$ and $0 < \alpha < \frac{n}{2}$.

We remark that equation (1.2) is equivalent to the following integral equation involving Riesz potentials,

$$(1.4) \quad u = \mathbf{I}_{2\alpha}(u^q d\sigma) + r, \quad u > 0.$$

It was shown in [CV14b] that if there exists a nontrivial solution u to the integral inequality

$$(1.5) \quad u \geq \mathbf{I}_{2\alpha}(u^q d\sigma) \quad \text{in } \mathbb{R}^n, \quad u \in L_{\text{loc}}^q(\mathbb{R}^n, d\sigma),$$

then u satisfies the lower bound

$$(1.6) \quad u(x) \geq c(\mathbf{I}_{2\alpha} \sigma(x))^{\frac{1}{1-q}}, \quad x \in \mathbb{R}^n,$$

where $c = c(n, q) > 0$. Therefore, a necessary condition for the existence of a solution to (1.4) is obviously that $\mathbf{I}_{2\alpha} \sigma \not\equiv \infty$, or equivalently,

$$(1.7) \quad \int_1^\infty \frac{\sigma(B(0, t))}{t^{n-2\alpha}} \frac{dt}{t} < +\infty.$$

Here and below $B(x, t)$ denotes a ball of radius t centered at $x \in \mathbb{R}^n$.

We will see that in fact $c_0 (\mathbf{I}_{2\alpha}\sigma)^{\frac{1}{1-q}}$ is a subsolution to (1.4) provided the constant $c_0 > 0$ is small enough, and (1.7) holds. In other words, in order to establish the existence of a solution to (1.4), it suffices to find a supersolution, which is the main goal of this paper.

It was also noticed in [CV14b] that if there exists a solution to (1.5), then σ must be absolutely continuous with respect to $\text{cap}_{\alpha,2}$, which is the $(\alpha, 2)$ -capacity defined by (2.1). In some of our results we will assume that σ satisfies the following stronger condition

$$(1.8) \quad \sigma(E) \leq c(\sigma) \text{cap}_{\alpha,2}(E), \quad \text{for all compact sets } E \subset \mathbb{R}^n,$$

where $c(\sigma) > 0$ is a constant which does not depend on E . Capacity conditions of this type were introduced and applied to various problems by V. Maz'ya (see [AH96], [Maz11], [Ver99]). We will show below (see Lemma 2.1) that this condition is equivalent, for every $s > 0$, to

$$(1.9) \quad \int_E (\mathbf{I}_{2\alpha}\sigma_E)^s d\sigma \leq c\sigma(E),$$

where σ_E denotes the restriction of σ to a compact set E . This implies, in particular, that for all balls B , or cubes in place of balls,

$$(1.10) \quad \int_B (\mathbf{I}_{2\alpha}\sigma_B)^s d\sigma \leq c\sigma(B).$$

Estimates of this type for $s \geq 1$ were considered in [Maz11], [Ver99], [JV10], [JV12], and for $s > \frac{1}{2}$ in [NTV03]. Such estimates for all $s > 0$ are new, and allow us to find a supersolution and obtain upper estimates of solutions to (1.4).

Condition (1.8), or equivalently (1.10) on cubes for some $s > 0$, together with (1.7), ensures the existence of a solution u to (1.4) which satisfies the following two-sided estimates if $r > 0$:

$$(1.11) \quad c^{-1}(r + \mathbf{I}_{2\alpha}\sigma)^{\frac{1}{1-q}} \leq u \leq c(r + \mathbf{I}_{2\alpha}\sigma)^{\frac{1}{1-q}},$$

where $c = c(n, q, r, c(\sigma)) > 0$. It also yields the existence of a solution u to the ground state problem (1.4) with $r = 0$ such that

$$(1.12) \quad c^{-1}(\mathbf{I}_{2\alpha}\sigma)^{\frac{1}{1-q}} \leq u \leq c\left(\mathbf{I}_{2\alpha}\sigma + (\mathbf{I}_2\sigma)^{\frac{1}{1-q}}\right).$$

These estimates are sharp as indicated in [BK92]. Indeed, suppose that $\mathbf{I}_{2\alpha}\sigma \in L^\infty(\mathbb{R}^n)$ as in [BK92]. Then by a well known result (see [AH96], [Maz11]) this implies that (1.8) holds. Therefore, there exists a solution u to (1.4) with $r = 0$ satisfying (1.12). Since $\mathbf{I}_{2\alpha}\sigma \in L^\infty(\mathbb{R}^n)$, this yields

$$c^{-1}(\mathbf{I}_{2\alpha}\sigma)^{\frac{1}{1-q}} \leq u \leq c\mathbf{I}_{2\alpha}\sigma,$$

which coincides with the Brezis–Kamin estimate (1.3) in the case $\alpha = 1$. However, condition (1.8) with $\alpha = 1$ is in general weaker than the condition $\mathbf{I}_2\sigma \in L^\infty(\mathbb{R}^n)$ imposed in [BK92], and is applicable to singular (unbounded) solutions.

Obviously, conditions (1.8) and (1.10) do not depend on the “sub-critical” growth rate q at all. As we will demonstrate below, condition (1.10) with $s = \frac{q}{1-q}$ can be relaxed to the following condition which does depend on q :

$$(1.13) \quad \int_{B(x, \frac{r}{2})} (\mathbf{I}_{2\alpha}\sigma)^{\frac{q}{1-q}} d\sigma \leq c \sigma(B(x, r)) \left(1 + \int_r^\infty \frac{\sigma(B(x, t))}{t^{n-2\alpha}} \frac{dt}{t}\right)^{\frac{q}{1-q}}.$$

It is easy to see that (1.13) yields the following pointwise condition on σ :

$$(1.14) \quad \mathbf{I}_{2\alpha} \left((\mathbf{I}_{2\alpha}\sigma)^{\frac{q}{1-q}} d\sigma \right) \leq \kappa \left(\mathbf{I}_{2\alpha}\sigma + (\mathbf{I}_{2\alpha}\sigma)^{\frac{1}{1-q}} \right),$$

where $\kappa = \kappa(\alpha, n, q, c) > 0$.

As we will demonstrate below, condition (1.14) combined with (1.7), is actually *necessary and sufficient* for the existence of a solution u to (1.4) with $r = 0$ satisfying the Brezis–Kamin type estimates (1.12).

Both conditions (1.13) and (1.14) are weaker than the capacity condition (1.8). It is easy to find $\sigma > 0$ such that (1.13) and hence (1.14) hold, but (1.8) fails. For instance, one can let $\sigma(x) = \frac{1}{|x|^s}$ on the ball $B(0, 1)$ and zero outside $B(0, 1)$, where $2\alpha < s < n - (n - 2\alpha)q$.

In fact, when σ is radially symmetric, we will characterize condition (1.14) as follows (see Proposition 5.1 in Sec. 5 below):

$$(1.15) \quad \limsup_{|x| \rightarrow 0} \frac{\frac{1}{|x|^{(n-2\alpha)(1-q)}} \int_{|y| < |x|} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}}}{\int_{|y| \geq |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}}} < \infty.$$

Also, for radially symmetric σ , we obtain necessary and sufficient conditions for the existence of a solution u to (1.4) with $r = 0$ in the following form:

$$(1.16) \quad \int_{|y| < 1} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}} < \infty \text{ and } \int_{|y| \geq 1} \frac{d\sigma(y)}{|y|^{n-2\alpha}} < \infty.$$

Moreover, such a solution u satisfies matching lower and upper estimates (see Theorem 5.1 below):

$$(1.17) \quad u(x) \approx \frac{1}{|x|^{n-2\alpha}} \left(\int_{|y| < |x|} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}} \right)^{\frac{1}{1-q}} + \left(\int_{|y| \geq |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \right)^{\frac{1}{1-q}}.$$

Here $A \approx B$ means $c_1 A \leq B \leq c_2 A$, where c_1 and c_2 are positive constants.

Before stating our main results for the quasilinear problem (1.1), we need to discuss assumptions analogous to (1.7) and (1.8) for the p -Laplacian. The Wolff potential $\mathbf{W}_{\alpha,p}\sigma$ is defined ([HW83]), for $1 < p < \infty$ and $0 < \alpha < \frac{n}{p}$, by

$$\mathbf{W}_{\alpha,p}\sigma(x) = \int_0^\infty \left(\frac{\sigma(B(x, t))}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}.$$

Let $\alpha = 1$, and suppose that $\mathbf{W}_{1,p}\sigma \not\equiv \infty$, or equivalently,

$$(1.18) \quad \int_1^\infty \left(\frac{\sigma(B(0,t))}{t^{n-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} < \infty.$$

Suppose additionally that σ satisfies the capacity condition

$$(1.19) \quad \sigma(E) \leq C(\sigma) \operatorname{cap}_p(E) \text{ for all compact sets } E \subset \mathbb{R}^n,$$

where $C(\sigma) > 0$ and $\operatorname{cap}_p(\cdot)$ stands for the p -capacity defined by

$$(1.20) \quad \operatorname{cap}_p(E) = \inf \{ \|\nabla f\|_{L^p}^p : f \geq 1 \text{ on } E, f \in C_0^\infty(\mathbb{R}^n) \}, \quad E \subset \mathbb{R}^n,$$

for compact sets E .

We are now ready to state our first main result for equation (1.1) with $r > 0$.

Theorem 1.1. *Let $1 < p < n$, $0 < q < p - 1$, and $r > 0$. Let $\sigma \in M^+(\mathbb{R}^n)$. If both (1.18) and (1.19) hold, then there exists a distributional solution $u \in W_{\operatorname{loc}}^{1,p}(\mathbb{R}^n)$ to (1.1) such that*

$$c^{-1} \left(r + \mathbf{W}_{1,p}\sigma \right)^{\frac{p-1}{p-1-q}} \leq u \leq c \left(r + \mathbf{W}_{1,p}\sigma \right)^{\frac{p-1}{p-1-q}},$$

where $W_{\operatorname{loc}}^{1,p}(\mathbb{R}^n)$ is the usual local Sobolev space and $c > 0$ depends only on n, p, q, r , and $C(\sigma)$. If $p \geq n$, then there are no nontrivial solutions on \mathbb{R}^n .

We observe that in this case we have matching upper and lower estimates of the solution u .

Our next theorem is concerned with the ground state problem (1.1) with $r = 0$.

Theorem 1.2. *Let $1 < p < n$ and $0 < q < p - 1$. Let $\sigma \in M^+(\mathbb{R}^n)$. If both (1.18) and (1.19) hold, then there exists a minimal p -superharmonic solution $u \in W_{\operatorname{loc}}^{1,p}(\mathbb{R}^n)$ to (1.1) with $r = 0$ such that*

$$(1.21) \quad c^{-1} \left(\mathbf{W}_{1,p}\sigma \right)^{\frac{p-1}{p-1-q}} \leq u \leq c \left(\mathbf{W}_{1,p}\sigma + \left(\mathbf{W}_{1,p}\sigma \right)^{\frac{p-1}{p-1-q}} \right),$$

where $c = c(n, p, q, C(\sigma)) > 0$. In the case $p \geq n$ there are no nontrivial solutions on \mathbb{R}^n .

See [KM94], [HKM06] for the definition of p -superharmonic solutions, or equivalently locally renormalized solutions discussed in [BV03], [KKT09].

In the next theorem, we give a necessary and sufficient condition for the existence of a solution to (1.1) satisfying (1.21).

Theorem 1.3. *Let $1 < p < n$, $0 < q < p - 1$ and $\sigma \in M^+(\mathbb{R}^n)$. Then there exists a p -superharmonic solution u to (1.1) with $r = 0$ satisfying (1.21) if and only if there exists a constant $\kappa > 0$ such that*

$$(1.22) \quad \mathbf{W}_{1,p} \left(\left(\mathbf{W}_{1,p}\sigma \right)^{\frac{(p-1)q}{p-1-q}} d\sigma \right) \leq \kappa \left(\mathbf{W}_{1,p}\sigma + \left(\mathbf{W}_{1,p}\sigma \right)^{\frac{p-1}{p-1-q}} \right) < \infty \quad \text{a.e.}$$

The plan of the paper is as follows. In Section 2, we study the equivalence between the capacity condition (1.8) and condition (1.10) for all $s > 0$. In Section 3, we study integral equations closely related to (1.1). Section 4 is devoted to a proof of our main results regarding equation (1.1). In Section 5, we consider the case where σ is radially symmetric, and give an explicit condition for the existence of a radial solution to (1.2), together with matching bilateral pointwise estimates of such a solution. We also provide a proof of (1.15), and conclude with an example where there is a solution to (1.4), but condition (1.14) fails, and consequently the upper estimate of Brezis–Kamin type in (1.12) is no longer true.

2. CAPACITY CONDITION AND WOLFF POTENTIAL ESTIMATES

For $p > 1$, we define the (α, p) -capacity of a subset $E \subset \mathbb{R}^n$ by

$$(2.1) \quad \text{cap}_{\alpha,p}(E) = \inf \{ \|f\|_{L^p(\mathbb{R}^n)}^p : \mathbf{I}_\alpha f \geq 1 \text{ on } E, \quad f \in L_+^p(\mathbb{R}^n) \},$$

where $\mathbf{I}_\alpha \sigma$ is the Riesz potential of order α defined for $0 < \alpha < n$ by

$$(2.2) \quad \mathbf{I}_\alpha \sigma(x) = \int_0^\infty \frac{\sigma(B(x, r))}{r^{n-\alpha}} \frac{dr}{r}, \quad x \in \mathbb{R}^n.$$

Notice that $\text{cap}_{1,p}(E) \approx \text{cap}_p(E)$ for all compact sets E (see [AH96]).

Lemma 2.1. *Let $\sigma \in M^+(\mathbb{R}^n)$ and suppose that*

$$(2.3) \quad \sigma(E) \leq C \text{cap}_{\alpha,p}(E), \quad \text{for all compact sets } E \subset \mathbb{R}^n.$$

Then, for every $s > 0$, the following inequality holds:

$$(2.4) \quad \int_E (\mathbf{W}_{\alpha,p} \sigma_E)^s d\sigma \leq c \sigma(E), \quad \text{for all compact sets } E.$$

Conversely, if (2.4) holds for a given $s > 0$, then (2.3) holds; consequently, (2.4) holds for every $s > 0$.

Proof. Condition (2.3) is known to be equivalent to the non-capacitary condition (see [Maz11], [Ver99])

$$(2.5) \quad \int_{\mathbb{R}^n} (\mathbf{I}_\alpha \sigma_E)^{p'} dx \leq c \sigma(E),$$

for all compact sets $E \subset \mathbb{R}^n$. By Wolff's inequality ([AH96]), we obtain

$$(2.6) \quad \int_{\mathbb{R}^n} (\mathbf{I}_\alpha \sigma_E)^{p'} dx \geq c \int_{\mathbb{R}^n} \mathbf{W}_{\alpha,p} \sigma_E d\sigma_E = c \int_E \mathbf{W}_{\alpha,p} \sigma_E d\sigma.$$

Hence,

$$(2.7) \quad \int_E \mathbf{W}_{\alpha,p} \sigma_E d\sigma_E \leq c \sigma(E), \quad \text{for all compact sets } E.$$

If $0 < s < 1$, then clearly the preceding estimate, together with Hölder's inequality, yields (2.4). If $s > 1$, we will show that (2.4) holds by using a shifted dyadic lattice \mathcal{D}_t as in [COV04]. Let E be a compact set in \mathbb{R}^n . Then

we have the following estimate for the truncated Wolff potential $\mathbf{W}_{\alpha,p}^r \sigma_E$ (see [COV04]),

$$\mathbf{W}_{\alpha,p}^r \sigma_E(x) \leq c_1 r^{-n} \int_{|t| \leq cr} \sum_{Q \in \mathcal{D}} \left[\frac{\sigma_E(Q+t)}{|Q+t|^{1-\frac{\alpha p}{n}}} \right]^{\frac{1}{p-1}} \chi_{Q+t}(x) dt.$$

Raising both sides to the power s , integrating over E with respect to $d\sigma_E$, and using Minkowski's inequality, we obtain

$$\begin{aligned} & \int_E (\mathbf{W}_{\alpha,p}^r \sigma_E(x))^s d\sigma_E \\ & \leq c_1 \left(r^{-n} \int_{|t| \leq cr} \left(\int_E \left[\sum_{Q \in \mathcal{D}} \left[\frac{\sigma_E(Q+t)}{|Q+t|^{1-\frac{\alpha p}{n}}} \right]^{\frac{1}{p-1}} \chi_{Q+t}(x) \right]^s d\sigma_E \right)^{\frac{1}{s}} dt \right)^s. \end{aligned}$$

Applying Proposition 2.2 in [COV04], we have

$$\begin{aligned} & \int_E \left[\sum_{Q \in \mathcal{D}} \left[\frac{\sigma_E(Q+t)}{|Q+t|^{1-\frac{\alpha p}{n}}} \right]^{\frac{1}{p-1}} \chi_{Q+t}(x) \right]^s d\sigma_E \\ & \leq c \sum_{Q \in \mathcal{D}} \left[\frac{\sigma_E(Q+t)}{|Q+t|^{1-\frac{\alpha p}{n}}} \right]^{\frac{1}{p-1}} \sigma_E(Q+t) \\ & \quad \times \left(\frac{1}{\sigma_E(Q+t)} \sum_{R \subset Q} \left[\frac{\sigma_E(R+t)}{|R+t|^{1-\frac{\alpha p}{n}}} \right]^{\frac{1}{p-1}} \sigma_E(R+t) \right)^{s-1}. \end{aligned}$$

Using Lemma 4.7 in [JV10], we see that the last factor is uniformly bounded, and hence by the same lemma

$$\int_E \left[\sum_{Q \in \mathcal{D}} \left[\frac{\sigma_E(Q+t)}{|Q+t|^{1-\frac{\alpha p}{n}}} \right]^{\frac{1}{p-1}} \chi_{Q+t}(x) \right]^s d\sigma_E \leq c \sigma(E).$$

Hence,

$$\int_E (\mathbf{W}_{\alpha,p}^r \sigma_E)^s d\sigma \leq c_1 \left(r^{-n} \int_{|t| \leq cr} (c \sigma(E))^{\frac{1}{s}} dt \right)^s \leq c \sigma(E).$$

Letting $r \rightarrow \infty$ and using the Monotone Convergence Theorem, we deduce

$$\int_E (\mathbf{W}_{\alpha,p} \sigma_E)^s d\sigma \leq c \sigma(E),$$

where the constant $c > 0$ depends on α, p, n, s and C .

Conversely, suppose that (2.4) holds for a fixed $s > 0$. Let E be a compact set in \mathbb{R}^n and suppose that $\sigma(E) > 0$. We write

$$\sigma(E) = \int_E (\mathbf{W}_{\alpha,p} \sigma_E)^{-\beta} (\mathbf{W}_{\alpha,p} \sigma_E)^\beta d\sigma,$$

where $\beta > 0$ will be chosen later. Using Hölder's inequality, we have

$$\sigma(E) \leq \left(\int_E (\mathbf{W}_{\alpha,p} \sigma_E)^{-\beta r} d\sigma \right)^{\frac{1}{r}} \left(\int_E (\mathbf{W}_{\alpha,p} \sigma_E)^{\beta r'} d\sigma \right)^{\frac{1}{r'}},$$

where $r > 1$ and $r' = \frac{r}{r-1}$. Let us choose β and r such that $\beta r = p - 1$ and $\beta r' = s$; hence, $r = 1 + \frac{p-1}{s}$ and $\beta = \frac{s(p-1)}{s+p-1}$. Consequently,

$$\sigma(E) \leq \left(\int_E \frac{d\sigma_E}{(\mathbf{W}_{\alpha,p} \sigma_E)^{p-1}} \right)^{\frac{1}{r}} \left(\int_E (\mathbf{W}_{\alpha,p} \sigma_E)^s d\sigma \right)^{\frac{1}{r'}}.$$

Applying Theorem 1.11 in [Ver99], we have

$$\int_E \frac{d\sigma_E}{(\mathbf{W}_{\alpha,p} \sigma_E)^{p-1}} \leq c \operatorname{cap}_{\alpha,p}(E).$$

Combining this estimate and (2.4) yields

$$\sigma(E) \leq c (\operatorname{cap}_{\alpha,p}(E))^{\frac{1}{r}} (\sigma(E))^{\frac{1}{r'}}.$$

This proves $\sigma(E) \leq c \operatorname{cap}_{\alpha,p}(E)$. Consequently, arguing as at the beginning of the proof, we also see that (2.4) holds for any $s > 0$. This completes the proof of Lemma 2.1. \square

Remark 2.2. Suppose that (2.3) holds. Then from (2.4) it follows that, for any $s > 0$,

$$(2.8) \quad \int_Q (\mathbf{W}_{\alpha,p} \sigma_Q)^s d\sigma \leq c(\alpha, p, n, C, s) \sigma(Q),$$

for all cubes Q in \mathbb{R}^n .

It can be shown using some results of the second author on duality of discrete Littlewood–Paley spaces (see, e.g., [CO15], Lemma 4.5) that if (2.8) holds for some $s > 0$, then it also holds for any $s > 0$; consequently, (2.3) holds.

The fact that (2.8) does not depend on s was proved for $s > \frac{1}{2}$ by Nazarov, Treil and Volberg [NTV03] using the Bellman function method.

3. SOLUTIONS OF THE NONLINEAR INTEGRAL EQUATION

3.1. Inhomogeneous problems. Consider the equation

$$(3.1) \quad u = \mathbf{W}_{\alpha,p}(u^q d\sigma) + r, \quad u > 0,$$

where $r > 0$, and $u \in L_{\text{loc}}^q(\mathbb{R}^n, d\sigma)$. We recall that if $u \geq \mathbf{W}_{\alpha,p}(u^q d\sigma)$, then

$$(3.2) \quad u(x) \geq c (\mathbf{W}_{\alpha,p} \sigma(x))^{\frac{p-1}{p-1-q}}, \quad x \in \mathbb{R}^n,$$

where $c = c(n, p, q, \alpha) > 0$ (Theorem 3.4 in [CV14b]). Therefore, a necessary condition for the existence of a nontrivial solution to (3.1) is that $\mathbf{W}_{\alpha,p} \sigma \not\equiv \infty$, or equivalently,

$$(3.3) \quad \int_1^\infty \left(\frac{\sigma(B(0, t))}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} < \infty.$$

Theorem 3.1. *Let $r > 0$, $1 < p < n$, $0 < q < p - 1$, and $0 < \alpha < \frac{n}{p}$. Let $\sigma \in M^+(\mathbb{R}^n)$. Suppose that both (2.3) and (3.3) hold. Then there exists a solution u to (3.1) such that*

$$(3.4) \quad c^{-1} \left(r + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right) \leq u \leq c \left(r + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right),$$

where $c = c(n, p, q, \alpha, C, r)$. Moreover, $u \in L_{\text{loc}}^s(\mathbb{R}^n, d\sigma)$, for every $s > 0$.

Proof. Let $w = c_0 \left(r + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right)$, where $c_0 > 0$ is a sufficiently small constant. Using Lemma 3.2 in [CV14a], we see that w is a subsolution to (3.1). Let

$$v = c \left(r + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right),$$

where $c > 0$ is a large constant to be determined later. We will show that v is a supersolution of (3.1). First, we estimate

$$\begin{aligned} \mathbf{W}_{\alpha,p}(v^q d\sigma)(x) &= \int_0^\infty \left(\frac{\int_{B(x,t)} v^q d\sigma}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq c^{\frac{q}{p-1}} c_1 \left(r^{\frac{q}{p-1}} \mathbf{W}_{\alpha,p}\sigma + \int_0^\infty \left(\frac{\int_{B(x,t)} (\mathbf{W}_{\alpha,p}\sigma)^{\frac{(p-1)q}{p-1-q}} d\sigma}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \right). \end{aligned}$$

Letting $\beta = \frac{(p-1)q}{p-1-q}$, we next estimate

$$\begin{aligned} \int_{B(x,t)} (\mathbf{W}_{\alpha,p}\sigma)^\beta d\sigma &= \int_{B(x,t)} \left[\int_0^\infty \left(\frac{\sigma(B(y,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^\beta d\sigma(y) \\ &\leq c_1 \int_{B(x,t)} \left[\int_0^t \left(\frac{\sigma(B(y,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^\beta d\sigma(y) \\ &\quad + c_1 \int_{B(x,t)} \left[\int_t^\infty \left(\frac{\sigma(B(y,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^\beta d\sigma(y) = c_1 (I + II). \end{aligned}$$

For $y \in B(x,t)$ and $s \geq t$, we have $B(y,s) \subset B(x,2s)$. Hence,

$$\begin{aligned} II &\leq \int_{B(x,t)} \left[\int_t^\infty \left(\frac{\sigma(B(x,2s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^\beta d\sigma(y) \\ &= \sigma(B(x,t)) \left[\int_t^\infty \left(\frac{\sigma(B(x,2s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^\beta \\ &\leq c_1 \sigma(B(x,t)) [\mathbf{W}_{\alpha,p}\sigma(x)]^\beta, \end{aligned}$$

where $c_1 = c_1(\alpha, n, p, q)$. For $s \leq t$, we have $B(y, s) \subset B(x, 2t)$, so

$$\begin{aligned} I &= \int_{B(x,t)} \left[\int_0^t \left(\frac{\sigma(B(y,s) \cap B(x,2t))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^\beta d\sigma(y) \\ &\leq \int_{B(x,2t)} \left[\int_0^t \left(\frac{\sigma(B(y,s) \cap B(x,2t))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^\beta d\sigma(y) \\ &\leq \int_{B(x,2t)} [\mathbf{W}_{\alpha,p} \sigma_{B(x,2t)}]^\beta d\sigma. \end{aligned}$$

Using (2.8), we obtain

$$\int_{B(x,2t)} [\mathbf{W}_{\alpha,p} \sigma_{B(x,2t)}]^\beta d\sigma \leq c_2 \sigma(B(x,2t)),$$

where $c_2 = c_2(\alpha, n, p, q, C)$. Thus,

$$\int_{B(x,t)} (\mathbf{W}_{\alpha,p} \sigma)^\beta d\sigma \leq c_1 [\mathbf{W}_{\alpha,p} \sigma(x)]^\beta \sigma(B(x,t)) + c_2 \sigma(B(x,2t)).$$

Consequently,

$$\begin{aligned} \left[\int_{B(x,t)} (\mathbf{W}_{\alpha,p} \sigma)^\beta d\sigma \right]^{\frac{1}{p-1}} &\leq c_1 (\mathbf{W}_{\alpha,p} \sigma(x))^{\frac{q}{p-1-q}} [\sigma(B(x,t))]^{\frac{1}{p-1}} \\ &\quad + c_2 [\sigma(B(x,2t))]^{\frac{1}{p-1}}. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbf{W}_{\alpha,p}(v^q d\sigma) &\leq c^{\frac{q}{p-1}} \tilde{c} \left(r^{\frac{q}{p-1}} \mathbf{W}_{\alpha,p} \sigma + c_2 \mathbf{W}_{\alpha,p} \sigma + c_1 (\mathbf{W}_{\alpha,p} \sigma)^{\frac{p-1}{p-1-q}} \right) \\ &\leq c^{\frac{q}{p-1}} c_3 \left(r + (\mathbf{W}_{\alpha,p} \sigma)^{\frac{p-1}{p-1-q}} \right), \end{aligned}$$

where $c_3 = c_3(n, p, q, \alpha, C, r)$. Therefore, picking c large enough yields $v \geq r + \mathbf{W}_{\alpha,p}(v^q d\sigma)$ and $v \geq w$.

Given a supersolution v , and a subsolution w such that $w \leq v$, we use a standard iteration argument, and the Monotone Convergence Theorem, to deduce the existence of a nontrivial solution u to (3.1) such that $w \leq u \leq v$, which yields (3.4). By Lemma 2.1, we deduce that $u \in L_{\text{loc}}^s(\mathbb{R}^n, d\sigma)$ for every $s > 0$. This concludes the proof of Theorem 3.1. \square

3.2. Homogeneous problems. Having the same hypotheses as in Theorem 3.1, we obtain the following result for the homogeneous equation

$$(3.5) \quad u = \mathbf{W}_{\alpha,p}(u^q d\sigma), \quad u \geq 0,$$

where $u \in L_{\text{loc}}^q(\mathbb{R}^n, d\sigma)$.

Theorem 3.2. *Let $1 < p < \infty$, $0 < q < p - 1$, and $0 < \alpha < \frac{n}{p}$. Let $\sigma \in M^+(\mathbb{R}^n)$. Suppose that both (2.3) and (3.3) hold. Then there exists a solution u to (3.5) such that*

$$(3.6) \quad c^{-1} (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \leq u \leq c \left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right),$$

where $c = c(n, p, q, \alpha, C)$. Moreover, $u \in L_{\text{loc}}^s(\mathbb{R}^n, d\sigma)$, for every $s > 0$.

Proof. Let $w = c_0 (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}}$ with a sufficiently small constant $c_0 > 0$. Then as in the proof of Theorem 3.1, w is a subsolution to (3.5). Now, let

$$v = c_1 \left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right),$$

where c_1 is a large constant. Clearly, $w \leq v$. Arguing as in the proof of Theorem 3.1 and using (2.8), we show that v is a supersolution to (3.5). Therefore, as above, we see that there exists a nontrivial solution u to (3.5) such that (3.6) holds.

Given $s > 0$, then $u \in L_{\text{loc}}^s(\mathbb{R}^n, d\sigma)$ follows from (3.6) and Lemma 2.1. We also notice that $\liminf_{|x| \rightarrow \infty} u(x) = 0$ since $\liminf_{|x| \rightarrow \infty} \mathbf{W}_{\alpha,p}\sigma(x) = 0$ by Corollary 3.2 in [CV14b]. This completes the proof of Theorem 3.2. \square

Instead of the capacity condition (2.3), let us use now a weaker condition

$$(3.7) \quad \mathbf{W}_{\alpha,p} \left((\mathbf{W}_{\alpha,p}\sigma)^{\frac{(p-1)q}{p-1-q}} d\sigma \right) \leq \kappa \left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right) < \infty \text{ a.e.,}$$

where κ is a positive constant. Then we can still construct solutions to (3.5) satisfying the Brezis–Kamin type estimates, and show that (3.7) is actually necessary for the existence of such solutions.

Theorem 3.3. *Let $1 < p < \infty$, $0 < q < p - 1$, and $0 < \alpha < \frac{n}{p}$. If $\sigma \in M^+(\mathbb{R}^n)$ satisfies (3.7), then there exists a solution u to (3.5) such that (3.6) holds with a constant $c > 0$ depending only on α, n, p, q, κ .*

Conversely, suppose that there exists a nontrivial supersolution u to (3.5) such that (3.6) holds. Then (3.7) holds with $\kappa = \kappa(p, q, c)$.

Proof. Suppose that (3.7) holds. Let $w = c_0 (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}}$ with small constant c_0 ; then w is a subsolution to (3.5) as before. Let

$$v = c \left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right),$$

where $c > 0$ is a large constant. We estimate

$$\begin{aligned} \mathbf{W}_{\alpha,p}(v^q d\sigma) &= c^{\frac{q}{p-1}} \mathbf{W}_{\alpha,p} \left(\left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right)^q d\sigma \right) \\ &\leq a c^{\frac{q}{p-1}} \mathbf{W}_{\alpha,p} ((\mathbf{W}_{\alpha,p}\sigma)^q d\sigma) + a c^{\frac{q}{p-1}} \mathbf{W}_{\alpha,p} \left((\mathbf{W}_{\alpha,p}\sigma)^{\frac{(p-1)q}{p-1-q}} d\sigma \right) \\ &\leq a c^{\frac{q}{p-1}} \mathbf{W}_{\alpha,p} ((\mathbf{W}_{\alpha,p}\sigma)^q d\sigma) + a c^{\frac{q}{p-1}} \kappa \left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right), \end{aligned}$$

where $a = a(p, q)$. Next, we write

$$\mathbf{W}_{\alpha,p}((\mathbf{W}_{\alpha,p}\sigma)^q d\sigma)(x) = \int_0^\infty \left(\frac{\int_{B(x,t)} (\mathbf{W}_{\alpha,p}\sigma)^q d\sigma}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}.$$

Using Hölder's inequality and Young's inequality, we obtain

$$\begin{aligned} \int_{B(x,t)} (\mathbf{W}_{\alpha,p}\sigma)^q d\sigma &\leq \left(\int_{B(x,t)} (\mathbf{W}_{\alpha,p}\sigma)^{\frac{(p-1)q}{p-1-q}} d\sigma \right)^{\frac{p-1-q}{p-1}} [\sigma(B(x,t))]^{\frac{q}{p-1}} \\ &\leq \tilde{b} \left(\int_{B(x,t)} (\mathbf{W}_{\alpha,p}\sigma)^{\frac{(p-1)q}{p-1-q}} d\sigma + \sigma(B(x,t)) \right). \end{aligned}$$

Hence,

$$\begin{aligned} &\int_0^\infty \left(\frac{\int_{B(x,t)} (\mathbf{W}_{\alpha,p}\sigma)^q d\sigma}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq b \int_0^\infty \left(\frac{\int_{B(x,t)} (\mathbf{W}_{\alpha,p}\sigma)^{\frac{(p-1)q}{p-1-q}} d\sigma}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} + b \int_0^\infty \left(\frac{\sigma(B(x,t))}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &= b \mathbf{W}_{\alpha,p} \left((\mathbf{W}_{\alpha,p}\sigma)^{\frac{(p-1)q}{p-1-q}} d\sigma \right)(x) + b \mathbf{W}_{\alpha,p}\sigma(x), \end{aligned}$$

where $b = b(p, q)$. By (1.22), the last term is bounded by

$$\begin{aligned} &b\kappa \left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right)(x) + b \mathbf{W}_{\alpha,p}\sigma(x) \\ &= b(\kappa + 1) \mathbf{W}_{\alpha,p}\sigma + b\kappa (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}}. \end{aligned}$$

It follows,

$$\begin{aligned} \mathbf{W}_{\alpha,p}(v^q d\sigma) &\leq a c^{\frac{q}{p-1}} b(\kappa + 1) \mathbf{W}_{\alpha,p}\sigma + a c^{\frac{q}{p-1}} b\kappa (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \\ &\quad + a c^{\frac{q}{p-1}} \kappa \left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right) \\ &\leq c^{\frac{q}{p-1}} a(b\kappa + b + \kappa) (\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}}). \end{aligned}$$

If c is chosen so that $c \geq c^{\frac{q}{p-1}} a(b\kappa + b + \kappa)$ and $c \geq c_0$, then we obtain $v \geq w$ and $v \geq \mathbf{W}_{\alpha,p}(v^q d\sigma)$. Using iterations as above, and the Monotone Convergence Theorem, we deduce from this the existence of a solution u to the equation $u = \mathbf{W}_{\alpha,p}(u^q d\sigma)$ which satisfies (3.6).

Conversely, suppose that there exists a nontrivial supersolution u to (3.5) such that (3.6) holds. Then clearly by the lower bound in (3.6),

$$u \geq \mathbf{W}_{\alpha,p}(u^q d\sigma) \geq (c^{-1})^{\frac{q}{p-1}} \mathbf{W}_{\alpha,p} \left((\mathbf{W}_{\alpha,p}\sigma)^{\frac{(p-1)q}{p-1-q}} d\sigma \right).$$

Using now the upper estimate in (3.6), we deduce (3.7) with $\kappa = \kappa(p, q, c)$. This completes the proof of Theorem 3.3. \square

4. PROOFS OF THEOREM 1.1, THEOREM 1.2, AND THEOREM 1.3

Proof of Theorem 1.1. Let $v \in L_{\text{loc}}^{1+q}(\mathbb{R}^n, d\sigma)$ be a solution to the integral equation

$$(4.1) \quad v = K\mathbf{W}_{1,p}(v^q d\sigma) + r,$$

where K is the constant in Corollary 4.5 in [PV08] (see also [KM94]). Then $\liminf_{|x| \rightarrow \infty} v(x) = r$, and v satisfies

$$(4.2) \quad c^{-1} \left(r + (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}} \right) \leq v \leq c \left(r + (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}} \right).$$

The existence of such a v follows from Theorem 3.1 with some modifications in the constants. We have

$$\int_B \mathbf{W}_{1,p}(v^q d\sigma_B) v^q d\sigma \leq c \int_B v^{1+q} d\sigma < \infty,$$

for every ball B . By a local version of Wolff's inequality (see [AH96], Theorem 4.5.5), we see that $v^q d\sigma \in W_{\text{loc}}^{-1,p'}(\mathbb{R}^n)$.

We set $u_0 = r$ and $B_k = B(0, 2^k)$, where $k = 0, 1, 2, \dots$. We have $u_0^q d\sigma \in W_{\text{loc}}^{-1,p'}(B_k)$ since $u_0 \leq v$ and $v^q d\sigma \in W_{\text{loc}}^{-1,p'}(\mathbb{R}^n)$. Hence, there exists a unique p -superharmonic solution u_1^k to the equation

$$(4.3) \quad -\Delta_p u_1^k = \sigma u_0^q \text{ in } B_k, \quad u_1^k \geq r, \quad u_1^k - r \in W_0^{1,p}(B_k).$$

(See, e.g., Theorem 21.6 in [HKM06].) By Corollary 4.5 in [PV08], we have

$$u_1^k - r \leq K\mathbf{W}_{1,p}(u_0^q d\sigma).$$

Since $u_0 \leq v$, we get

$$u_1^k \leq K\mathbf{W}_{1,p}(v^q d\sigma) + r = v.$$

We see that the sequence $\{u_1^k\}_k$ is increasing by a comparison principle (Lemma 5.1 in [CV14b]). Letting $u_1 = \lim_{k \rightarrow \infty} u_1^k$ and using the weak continuity of the p -Laplacian ([TW02]) and the Monotone Convergence Theorem, we deduce that u_1 is a p -superharmonic solution to the equation

$$-\Delta_p u_1 = \sigma u_0^q \text{ in } \mathbb{R}^n.$$

Moreover, $r \leq u_1 \leq v$ since $u_1^k \leq v$ and hence $\liminf_{|x| \rightarrow \infty} u_1(x) = r$. Clearly, we also have $u_0 \leq u_1$. We notice that $u_0^q d\sigma \in W_{\text{loc}}^{-1,p'}(\mathbb{R}^n)$ since $u_0 \leq v$ and $v^q d\sigma \in W_{\text{loc}}^{-1,p'}(\mathbb{R}^n)$. Therefore, applying Lemma 3.3 in [CV14b], we conclude that $u_1 \in W_{\text{loc}}^{1,p}(\mathbb{R}^n)$.

Let us now construct by induction a sequence $\{u_j\}_j$ of p -superharmonic functions in \mathbb{R}^n , $u_j \in L_{\text{loc}}^q(\mathbb{R}^n, d\sigma)$, so that

$$(4.4) \quad \begin{cases} -\Delta_p u_j = \sigma u_{j-1}^q \text{ in } \mathbb{R}^n, & j = 2, 3, \dots, \\ r \leq u_j \leq v, & u_j \in W_{\text{loc}}^{1,p}(\mathbb{R}^n), \\ u_{j-1} \leq u_j, \\ \liminf_{|x| \rightarrow \infty} u_j(x) = r. \end{cases}$$

Suppose that u_1, \dots, u_{j-1} have been constructed. We see that $\sigma u_{j-1}^q \in W^{-1,p'}(B_k)$ since $u_{j-1} \leq v$ and $v^q d\sigma \in W_{\text{loc}}^{-1,p'}(\mathbb{R}^n)$. Thus, as before, there exists a unique p -superharmonic solution u_j^k to the equation

$$(4.5) \quad -\Delta_p u_j^k = \sigma u_{j-1}^q \text{ in } B_k, \quad u_j^k \geq r, \quad u_j^k - r \in W_0^{1,p}(B_k).$$

Arguing by induction, let u_{j-1}^k be the unique solution of the equation

$$(4.6) \quad -\Delta_p u_{j-1}^k = \sigma u_{j-2}^q \text{ in } B_k, \quad u_{j-1}^k \geq r, \quad u_{j-1}^k - r \in W_0^{1,p}(B_k).$$

Since $u_{j-2} \leq u_{j-1}$, by the comparison principle we deduce that

$$(4.7) \quad u_{j-1}^k \leq u_j^k, \quad \text{for all } k \geq 1.$$

Using Corollary 4.5 in [PV08], we have

$$u_j^k - r \leq K \mathbf{W}_{1,p}(u_{j-1}^q d\sigma).$$

Since $u_{j-1} \leq v$, we obtain

$$u_j^k \leq K \mathbf{W}_{1,p}(v^q d\sigma) + r = v.$$

Applying again the comparison principle, we see that the sequence $\{u_j^k\}_k$ is increasing. Hence, letting $u_j = \lim_{k \rightarrow \infty} u_j^k$ and using the weak continuity ([TW02]) and the Monotone Convergence Theorem, we see that u_j is a p -superharmonic solution to the equation

$$-\Delta_p u_j = \sigma u_{j-1}^q \text{ in } \mathbb{R}^n.$$

Moreover, $r \leq u_j \leq v$ since $u_j^k \leq v$ and hence $\liminf_{|x| \rightarrow \infty} u_j(x) = r$. We also have $u_{j-1} \leq u_j$ since $u_{j-1}^k \leq u_j^k$. We see that $u_{j-1}^q d\sigma \in W_{\text{loc}}^{-1,p'}(\mathbb{R}^n)$ since $u_{j-1} \leq v$ and $v^q d\sigma \in W_{\text{loc}}^{-1,p'}(\mathbb{R}^n)$. Therefore, by Lemma 3.3 in [CV14b], it follows that $u_j \in W_{\text{loc}}^{1,p}(\mathbb{R}^n)$. By the weak continuity ([TW02]) and the Monotone Convergence Theorem, we deduce that u is a solution to the equation

$$-\Delta_p u = \sigma u^q \text{ in } \mathbb{R}^n.$$

Furthermore, $r \leq u \leq v$, and hence $\liminf_{|x| \rightarrow \infty} u(x) = r$. By (4.2), we get

$$u \leq c \left(r + \mathbf{W}_{1,p} \sigma \right)^{\frac{p-1}{p-1-q}}.$$

Using Lemma 3.3 in [CV14b] again, we conclude that $u \in W_{\text{loc}}^{1,p}(\mathbb{R}^n)$ since $u^q d\sigma \in W_{\text{loc}}^{-1,p'}(\mathbb{R}^n)$. The lower estimate follows from (3.2) and the fact that $u \geq r$. This completes the proof of Theorem 1.1. \square

Proof of Theorem 1.2. Suppose that (1.18) and (1.19) hold. Then by Theorem 3.2 there exists a nontrivial solution $v \in L_{\text{loc}}^{1+q}(\mathbb{R}^n, d\sigma)$ to the equation

$$(4.8) \quad v = K \mathbf{W}_{1,p}(v^q d\sigma), \quad \liminf_{|x| \rightarrow \infty} v(x) = 0.$$

Moreover, there exists a constant $c = c(n, p, q, C(\sigma)) > 0$ such that

$$(4.9) \quad v \leq c \left(\mathbf{W}_{1,p} \sigma + (\mathbf{W}_{1,p} \sigma)^{\frac{p-1}{p-1-q}} \right).$$

Arguing as in the proof of Theorem 1.1 in [CV14b], we deduce the existence of a minimal p -superharmonic solution u to the equation

$$-\Delta_p u = \sigma u^q \text{ in } \mathbb{R}^n.$$

Since $u \leq v$, we see that $u \in L_{\text{loc}}^{1+q}(\mathbb{R}^n, d\sigma)$. Hence,

$$\int_B \mathbf{W}_{1,p}(u^q d\sigma_B) u^q d\sigma \leq c \int_B u^{1+q} d\sigma < \infty,$$

for every ball B . By the local Wolff's inequality, it follows that $u^q d\sigma \in W_{\text{loc}}^{-1,p'}(\mathbb{R}^n)$. Hence $u \in W_{\text{loc}}^{1,p}(\mathbb{R}^n)$ by Lemma 3.3 in [CV14b].

Moreover, by (3.2), $u \geq c (\mathbf{W}_{1,p} \sigma)^{\frac{p-1}{p-1-q}}$, and consequently,

$$c^{-1} (\mathbf{W}_{1,p} \sigma)^{\frac{p-1}{p-1-q}} \leq u \leq c \left(\mathbf{W}_{1,p} \sigma + (\mathbf{W}_{1,p} \sigma)^{\frac{p-1}{p-1-q}} \right).$$

The case $p \geq n$ follows from Theorem 2.4 (ii) in [CV14b]. This completes the proof of Theorem 1.2. \square

Proof of Theorem 1.3. Suppose there exists a p -superharmonic solution u to (1.1) with $r = 0$, and u satisfies (1.21). By Corollary 4.5 in [PV08], $u \geq \frac{1}{K} \mathbf{W}_{1,p}(u^q d\sigma)$. Consequently, by (3.2), $u \geq C (\mathbf{W}_{1,p} \sigma)^{\frac{p-1}{p-1-q}}$; and hence,

$$u \geq c \mathbf{W}_{1,p}((\mathbf{W}_{1,p} \sigma)^{\frac{(p-1)q}{p-1-q}} d\sigma).$$

Therefore,

$$\mathbf{W}_{1,p}((\mathbf{W}_{1,p} \sigma)^{\frac{(p-1)q}{p-1-q}} d\sigma) \leq c \left(\mathbf{W}_{1,p} \sigma + (\mathbf{W}_{1,p} \sigma)^{\frac{p-1}{p-1-q}} \right) < \infty \text{ a.e.}$$

Conversely, suppose that (1.22) holds, then applying Theorem 3.3 and arguing as in the proof of Theorem 1.1 in [CV14b], we conclude the proof of Theorem 1.3. \square

Remark 4.1. Since our approach is based on the Wolff potential estimates (see [KM94], [KuMi14], [Lab02], [TW02], [PV08]), all of the results mentioned above remain valid if one replaces the p -Laplacian Δ_p in the model problem (1.1) by a more general quasilinear operator $\text{div} \mathcal{A}(x, \nabla \cdot)$, under standard structural assumptions on $\mathcal{A}(x, \xi)$ which ensure that $\mathcal{A}(x, \xi) \cdot \xi \approx |\xi|^p$, or a fully nonlinear operator of k -Hessian type (see details in [CV14b]).

5. THE RADIAL CASE

In this section we will assume that $\sigma \in M^+(\mathbb{R}^n)$ is radially symmetric. Suppose $0 < q < 1$ and $0 < 2\alpha < n$. We study equation (1.2) with $r = 0$, i.e.,

$$(5.1) \quad \begin{cases} (-\Delta)^\alpha u = \sigma u^q & \text{in } \mathbb{R}^n, \\ \liminf_{|x| \rightarrow \infty} u(x) = 0. \end{cases}$$

We notice that a necessary condition for the existence of a nontrivial solution to (5.1) is that σ must be absolutely continuous with respect to the $(\alpha, 2)$ -capacity $\text{cap}_{\alpha, 2}(\cdot)$ [CV14b]. In particular, σ has no atoms. Suppose that σ is radial and $\mathbf{I}_{2\alpha}\sigma \not\equiv +\infty$, then we have ([Rub96], p. 231)

$$\begin{aligned} \mathbf{I}_{2\alpha}\sigma(x) &= c \int_{\mathbb{R}^n} \int_{\max\{|x|, |y|\}}^{\infty} (|t|^2 - |x|^2)^{\alpha-1} (|t|^2 - |y|^2)^{\alpha-1} \frac{dt}{t^{2\alpha+n-3}} d\sigma(y) \\ &\approx \frac{\sigma(B(0, |x|))}{|x|^{n-2\alpha}} + \int_{|y| \geq |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}}, \end{aligned}$$

where we drop the first term if $x = 0$. We have the following theorem.

Theorem 5.1. *Let $0 < q < 1$, $n \geq 1$, and $0 < \alpha < \frac{n}{2}$. Let $\sigma \in M^+(\mathbb{R}^n)$ be radially symmetric. Then there exists a nontrivial solution u to (5.1) if and only if (1.16) holds. Moreover, u satisfies (1.17).*

Proof of Theorem 5.1. We remark that (5.1) is understood in the sense that $u = \mathbf{I}_{2\alpha}(u^q d\sigma)$. Suppose that u is a solution to (5.1). We first notice that $\mathbf{I}_{2\alpha}\sigma$ is radial. Therefore, the minimal solution to (5.1) constructed in Theorem 4.8 in [CV14b] is radial as well. Hence, we may assume that u is radial. By (3.2),

$$u(x) \geq c (\mathbf{I}_{2\alpha}\sigma(x))^{\frac{1}{1-q}}, \quad x \in \mathbb{R}^n,$$

where $c = c(n, q) > 0$. Consequently,

$$u(x) \geq c \left(\int_{|y| \geq |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \right)^{\frac{1}{1-q}}, \quad x \in \mathbb{R}^n.$$

We have

$$(5.2) \quad u(x) = \mathbf{I}_{2\alpha}(u^q d\sigma)(x) \approx \frac{\int_{|y| < |x|} u^q d\sigma}{|x|^{n-2\alpha}} + \int_{|y| \geq |x|} \frac{u^q d\sigma(y)}{|y|^{n-2\alpha}}.$$

By Lemma 4.2 in [CV14b], we have, for all $\nu \in M^+(\mathbb{R}^n)$,

$$\|\mathbf{I}_{2\alpha}\nu\|_{L^q(d\sigma_{B(0, |x|)})} \leq c \left(\int_{B(0, |x|)} u^q d\sigma \right)^{1-q} \nu(\mathbb{R}^n).$$

Let $\nu = \delta_0$, we get

$$\left(\int_{B(0, |x|)} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}} \right)^{\frac{1}{1-q}} \leq c \int_{B(0, |x|)} u^q d\sigma.$$

Therefore, we deduce from (5.2)

$$u(x) \geq c \frac{1}{|x|^{n-2\alpha}} \left(\int_{B(0, |x|)} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}} \right)^{\frac{1}{1-q}}.$$

Thus, (1.16) follows since $u \not\equiv +\infty$.

Conversely, suppose that condition (1.16) holds. This implies

$$\int_{B(0,|x|)} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}} < \infty \text{ and } \int_{|y|\geq|x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} < \infty, \quad x \neq 0.$$

Let $u_0 = c_0 (\mathbf{I}_{2\alpha}\sigma)^{\frac{1}{1-q}}$ with a small constant c_0 . Then $u_0 \leq \mathbf{I}_{2\alpha}(u_0^q d\sigma)$ as before.

$$\text{Let } v(x) = c \left(\frac{1}{|x|^{n-2\alpha}} \left(\int_{|y|<|x|} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}} \right)^{\frac{1}{1-q}} + \left(\int_{|y|\geq|x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \right)^{\frac{1}{1-q}} \right).$$

We see that $v \geq \mathbf{I}_{2\alpha}(v^q d\sigma)$. Indeed, for $x \neq 0$, we have

$$\begin{aligned} \mathbf{I}_{2\alpha}(v^q d\sigma)(x) &\approx \frac{1}{|x|^{n-2\alpha}} \int_{|y|<|x|} v^q d\sigma + \int_{|y|\geq|x|} \frac{v^q d\sigma}{|y|^{n-2\alpha}} \\ &\leq c^q \frac{1}{|x|^{n-2\alpha}} \int_{|y|<|x|} \frac{1}{|y|^{(n-2\alpha)q}} \left(\int_{|z|<|y|} \frac{d\sigma(z)}{|z|^{(n-2\alpha)q}} \right)^{\frac{q}{1-q}} d\sigma(y) \\ &\quad + c^q \frac{1}{|x|^{n-2\alpha}} \int_{|y|<|x|} \left(\int_{|z|\geq|y|} \frac{d\sigma(z)}{|z|^{n-2\alpha}} \right)^{\frac{q}{1-q}} d\sigma(y) \\ &\quad + c^q \int_{|y|\geq|x|} \frac{1}{|y|^{n-2\alpha}} \frac{1}{|y|^{(n-2\alpha)q}} \left(\int_{|z|<|y|} \frac{d\sigma(z)}{|z|^{(n-2\alpha)q}} \right)^{\frac{q}{1-q}} d\sigma(y) \\ &\quad + c^q \int_{|y|\geq|x|} \frac{1}{|y|^{n-2\alpha}} \left(\int_{|z|\geq|y|} \frac{d\sigma(z)}{|z|^{n-2\alpha}} \right)^{\frac{q}{1-q}} d\sigma(y) \\ &:= c^q (I + II + III + IV). \end{aligned}$$

Clearly,

$$I \leq \frac{1}{|x|^{n-2\alpha}} \left(\int_{|y|<|x|} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}} \right)^{\frac{1}{1-q}}.$$

We next estimate

$$\begin{aligned} II &= \frac{1}{|x|^{n-2\alpha}} \int_{|y|<|x|} \left(\int_{|y|\leq|z|<|x|} \frac{d\sigma(z)}{|z|^{n-2\alpha}} + \int_{|z|\geq|x|} \frac{d\sigma(z)}{|z|^{n-2\alpha}} \right)^{\frac{q}{1-q}} d\sigma(y) \\ &\leq c_1 \frac{1}{|x|^{n-2\alpha}} \int_{|y|<|x|} \left(\int_{|y|\leq|z|<|x|} \frac{d\sigma(z)}{|z|^{n-2\alpha}} \right)^{\frac{q}{1-q}} d\sigma(y) \\ &\quad + c_1 \frac{1}{|x|^{n-2\alpha}} \int_{|y|<|x|} d\sigma(y) \left(\int_{|z|\geq|x|} \frac{d\sigma(z)}{|z|^{n-2\alpha}} \right)^{\frac{q}{1-q}} \\ &= c_1 (II_a + II_b). \end{aligned}$$

Next,

$$\begin{aligned}
II_a &= \frac{1}{|x|^{n-2\alpha}} \int_{|y| < |x|} \left(\int_{|y| \leq |z| < |x|} \frac{d\sigma(z)}{|z|^{(n-2\alpha)q} |z|^{(n-2\alpha)(1-q)}} \right)^{\frac{q}{1-q}} d\sigma(y) \\
&\leq \frac{1}{|x|^{n-2\alpha}} \int_{|y| < |x|} \left(\int_{|y| \leq |z| < |x|} \frac{d\sigma(z)}{|z|^{(n-2\alpha)q}} \right)^{\frac{q}{1-q}} \frac{1}{|y|^{(n-2\alpha)q}} d\sigma(y) \\
&\leq \frac{1}{|x|^{n-2\alpha}} \left(\int_{|y| < |x|} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}} \right)^{\frac{1}{1-q}}.
\end{aligned}$$

Using Young's inequality with exponents $\frac{1}{1-q}$ and $\frac{1}{q}$, we obtain

$$II_b \leq c_2 \left(\left(\frac{1}{|x|^{n-2\alpha}} \int_{|y| < |x|} d\sigma(y) \right)^{\frac{1}{1-q}} + \left(\int_{|z| \geq |x|} \frac{d\sigma(z)}{|z|^{n-2\alpha}} \right)^{\frac{1}{1-q}} \right).$$

We next estimate

$$\begin{aligned}
III &\leq c_1 \int_{|y| \geq |x|} \frac{1}{|y|^{n-2\alpha} |y|^{(n-2\alpha)q}} \left(\int_{|z| < |x|} \frac{d\sigma(z)}{|z|^{(n-2\alpha)q}} \right)^{\frac{q}{1-q}} d\sigma(y) \\
&+ c_1 \int_{|y| \geq |x|} \frac{1}{|y|^{n-2\alpha} |y|^{(n-2\alpha)q}} \left(\int_{|x| \leq |z| < |y|} \frac{d\sigma(z)}{|z|^{(n-2\alpha)q}} \right)^{\frac{q}{1-q}} d\sigma(y) \\
&\leq c_1 \frac{1}{|x|^{(n-2\alpha)q}} \left(\int_{|z| < |x|} \frac{d\sigma(z)}{|z|^{(n-2\alpha)q}} \right)^{\frac{q}{1-q}} \int_{|y| \geq |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \\
&+ c_1 \int_{|y| \geq |x|} \frac{1}{|y|^{n-2\alpha} |y|^{(n-2\alpha)q}} \left(\int_{|x| \leq |z| < |y|} \frac{|z|^{(n-2\alpha)(1-q)} d\sigma(z)}{|z|^{n-2\alpha}} \right)^{\frac{q}{1-q}} d\sigma(y) \\
&\leq c_1 \frac{1}{|x|^{(n-2\alpha)q}} \left(\int_{|z| < |x|} \frac{d\sigma(z)}{|z|^{(n-2\alpha)q}} \right)^{\frac{q}{1-q}} \int_{|y| \geq |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \\
&+ c_1 \left(\int_{|x| \leq |z|} \frac{d\sigma(z)}{|z|^{n-2\alpha}} \right)^{\frac{q}{1-q}} \int_{|y| \geq |x|} \frac{d\sigma(y)}{|y|^{(n-2\alpha)}}.
\end{aligned}$$

Using Young's inequality again, we arrive at

$$\begin{aligned}
III &\leq c_1 c_2 \frac{1}{|x|^{n-2\alpha}} \left(\int_{|z| < |x|} \frac{d\sigma(z)}{|z|^{(n-2\alpha)q}} \right)^{\frac{1}{1-q}} \\
&+ (c_1 c_2 + c_1) \left(\int_{|y| \geq |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \right)^{\frac{1}{1-q}}.
\end{aligned}$$

Clearly,

$$IV \leq \left(\int_{|y| \geq |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \right)^{\frac{1}{1-q}}.$$

Clearly, $v(0) \geq \mathbf{I}_{2\alpha}(v^q d\sigma)(0)$, if c is chosen large enough. Therefore, we obtain $\mathbf{I}_{2\alpha}(v^q d\sigma) \leq v$. Using iterations and the Monotone Convergence Theorem as above, we deduce the existence of a radial solution u to the equation $u = \mathbf{I}_{2\alpha}(u^q d\sigma)$. Moreover,

$$u(x) \leq c \left(\frac{1}{|x|^{n-2\alpha}} \left(\int_{|y| < |x|} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}} \right)^{\frac{1}{1-q}} + \left(\int_{|y| \geq |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \right)^{\frac{1}{1-q}} \right),$$

which completes the proof of Theorem 5.1. \square

We now characterize condition (1.14) when σ is radial and $\mathbf{I}_{2\alpha}\sigma \not\equiv \infty$. Then for every $a > 0$,

$$\mathbf{I}_{2\alpha}(x) \leq c(a, \sigma), \quad \text{for } |x| \geq a.$$

Indeed,

$$\begin{aligned} \mathbf{I}_{2\alpha}\sigma(x) &\approx \frac{\sigma(B(0, |x|))}{|x|^{n-2\alpha}} + \int_{|y| \geq |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \\ &= \frac{\int_{|y| < a} d\sigma(y)}{|x|^{n-2\alpha}} + \frac{\int_{a \leq |y| < |x|} d\sigma(y)}{|x|^{n-2\alpha}} + \int_{|y| \geq |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \\ &\leq \frac{\int_{|y| < a} d\sigma(y)}{|a|^{n-2\alpha}} + \int_{a \leq |y| < |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} + \int_{|y| \geq a} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \\ &\leq \frac{\int_{|y| < a} d\sigma(y)}{|a|^{n-2\alpha}} + 2 \int_{|y| \geq a} \frac{d\sigma(y)}{|y|^{n-2\alpha}}. \end{aligned}$$

We observe that $\lim_{|x| \rightarrow \infty} \mathbf{I}_{2\alpha}\sigma(x) = 0$. If $\limsup_{|x| \rightarrow 0} \mathbf{I}_{2\alpha}\sigma(x) < +\infty$, then by the above observation, we have $\mathbf{I}_{2\alpha}\sigma \in L^\infty(\mathbb{R}^n)$. This implies that condition (1.14) holds. Therefore, we need to focus on the case where $\limsup_{|x| \rightarrow 0} \mathbf{I}_{2\alpha}\sigma(x) = +\infty$. Let us set

$$\mathbf{K}\sigma(x) = \frac{1}{|x|^{n-2\alpha}} \left(\int_{|y| < |x|} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}} \right)^{\frac{1}{1-q}}, \quad x \neq 0.$$

Suppose that (1.14) holds. Then by Theorem 1.2, there exists a solution u to (5.1) such that $u \leq c(\mathbf{I}_{2\alpha}\sigma + (\mathbf{I}_{2\alpha}\sigma)^{\frac{1}{1-q}})$. On the other hand, $u \geq c \mathbf{K}\sigma$ by Theorem 5.1. Therefore, if $\mathbf{I}_{2\alpha}\sigma \not\equiv \infty$, then condition (1.14) implies

$$(5.3) \quad \mathbf{K}\sigma \leq c \left(\mathbf{I}_{2\alpha}\sigma + (\mathbf{I}_{2\alpha}\sigma)^{\frac{1}{1-q}} \right) < \infty \quad \text{a.e.}$$

Conversely, suppose that (5.3) holds. Then by Theorem 5.1 there exists a solution u to (5.1) such that $u \leq c \left(\mathbf{K}\sigma + (\mathbf{I}_{2\alpha}\sigma)^{\frac{1}{1-q}} \right)$. Hence, $u \leq$

$c \left(\mathbf{I}_{2\alpha}\sigma + (\mathbf{I}_{2\alpha}\sigma)^{\frac{1}{1-q}} \right)$, and using (1.6) yields (1.14). Therefore, if $\mathbf{I}_{2\alpha}\sigma \not\equiv \infty$, then (1.14) holds if and only if (5.3) holds.

We next prove the following proposition.

Proposition 5.2. *Let $0 < q < 1$, $n \geq 1$, and $0 < 2\alpha < n$. Let $\sigma \in M^+(\mathbb{R}^n)$ be radially symmetric. Suppose that $\limsup_{|x| \rightarrow 0} \mathbf{I}_{2\alpha}\sigma(x) = +\infty$. Then there exists a constant $c > 0$ such that (5.3) holds if and only if $\int_{|y| \geq 1} \frac{d\sigma(y)}{|y|^{n-2\alpha}} < \infty$ and*

$$(5.4) \quad \limsup_{|x| \rightarrow 0} \frac{\frac{1}{|x|^{(n-2\alpha)(1-q)}} \int_{B(0,|x|)} \frac{d\sigma(y)}{|y|^{q(n-2\alpha)}}}{\int_{|y| \geq |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}}} < \infty.$$

Proof. Suppose that $\int_{|y| \geq 1} \frac{d\sigma(y)}{|y|^{n-2\alpha}} < \infty$ and that (5.4) holds. Then there exists δ , $0 < \delta < 1$, such that

$$\mathbf{K}\sigma(x) \leq c (\mathbf{I}_{2\alpha}\sigma(x))^{\frac{1}{1-q}}, \quad \text{for all } 0 < |x| < \delta.$$

For $\delta \leq |x| \leq 1$, we have

$$\frac{1}{|x|^{n-2\alpha}} \left(\int_{B(0,|x|)} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}} \right)^{\frac{1}{1-q}} \leq \frac{1}{\delta^{n-2\alpha}} \left(\int_{B(0,1)} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}} \right)^{\frac{1}{1-q}}.$$

On the other hand,

$$\begin{aligned} \mathbf{I}_{2\alpha}\sigma(x) &\approx \frac{1}{|x|^{n-2\alpha}} \int_{B(0,|x|)} d\sigma(y) + \int_{|y| \geq |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \\ &\geq \int_{B(0,\delta)} d\sigma(y) + \int_{|y| \geq 1} \frac{d\sigma(y)}{|y|^{n-2\alpha}}. \end{aligned}$$

Therefore, there exists a constant $c = c(\delta, \sigma)$ such that

$$\mathbf{K}\sigma(x) \leq c \mathbf{I}_{2\alpha}\sigma(x), \quad \text{when } \delta \leq |x| \leq 1.$$

For $|x| \geq 1$, we have

$$\int_{B(0,|x|)} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}} \leq \int_{0 \leq |y| \leq 1} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}} + \int_{1 \leq |y| < |x|} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}}.$$

By Hölder's inequality,

$$\int_{1 \leq |y| < |x|} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}} \leq \left(\int_{1 \leq |y| < |x|} d\sigma(y) \right)^{1-q} \left(\int_{1 \leq |y| < |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \right)^q.$$

Hence,

$$\begin{aligned} \left(\int_{1 \leq |y| < |x|} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}} \right)^{\frac{1}{1-q}} &\leq \left(\int_{1 \leq |y| < |x|} d\sigma(y) \right) \left(\int_{1 \leq |y| < |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \right)^{\frac{q}{1-q}} \\ &\leq \left(\int_{|y| < |x|} d\sigma(y) \right) \left(\int_{|y| \geq 1} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \right)^{\frac{q}{1-q}}. \end{aligned}$$

Therefore, there exists a constant $c = c(\sigma, q)$ such that

$$\mathbf{K}\sigma(x) \leq c \mathbf{I}_{2\alpha}\sigma(x), \quad \text{for } |x| \geq 1.$$

From this it follows

$$\mathbf{K}\sigma(x) \leq c(\mathbf{I}_{2\alpha}\sigma(x) + (\mathbf{I}_{2\alpha}\sigma(x))^{\frac{1}{1-q}}) < \infty, \quad x \neq 0.$$

Conversely, suppose that (5.3) holds and $\limsup_{|x| \rightarrow 0} \mathbf{I}_{2\alpha}\sigma(x) = +\infty$. Then for $|x|$ small enough we have

$$\mathbf{I}_{2\alpha}\sigma(x) \leq (\mathbf{I}_{2\alpha}\sigma(x))^{\frac{1}{1-q}}.$$

Consequently,

$$\mathbf{K}\sigma(x) \leq c(\mathbf{I}_{2\alpha}\sigma(x))^{\frac{1}{1-q}},$$

when $|x|$ is close to 0, which yields

$$(5.5) \quad \begin{aligned} & \frac{1}{|x|^{(n-2\alpha)(1-q)}} \int_{|y| < |x|} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}} \\ & \leq c \left(\frac{1}{|x|^{n-2\alpha}} \int_{B(0, |x|)} d\sigma(y) + \int_{|y| \geq |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \right). \end{aligned}$$

We estimate

$$\begin{aligned} \frac{1}{|x|^{n-2\alpha}} \int_{B(0, |x|)} d\sigma(y) &= \frac{1}{|x|^{(n-2\alpha)(1-q)} |x|^{(n-2\alpha)q}} \int_{0 \leq |y| \leq \delta} \frac{|y|^{(n-2\alpha)q} d\sigma(y)}{|y|^{(n-2\alpha)q}} \\ &+ \frac{1}{|x|^{n-2\alpha}} \int_{\delta < |y| < |x|} \frac{|y|^{n-2\alpha} d\sigma(y)}{|y|^{n-2\alpha}} \\ &\leq \frac{\delta^{(n-2\alpha)q}}{|x|^{(n-2\alpha)(1-q)} |x|^{(n-2\alpha)q}} \int_{0 \leq |y| \leq \delta} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}} \\ &+ \int_{\delta < |y| < |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}}. \end{aligned}$$

Letting $\delta = c_1 |x|$ where c_1 is small enough, we obtain

$$\begin{aligned} \frac{1}{|x|^{n-2\alpha}} \int_{B(0, |x|)} d\sigma(y) &\leq c_1^{(n-2\alpha)q} \frac{1}{|x|^{(n-2\alpha)(1-q)}} \int_{0 \leq |y| < |x|} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}} \\ &+ \int_{|y| > c_1 |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}}. \end{aligned}$$

Hence, by (5.5), we get

$$\begin{aligned} & \frac{1}{|x|^{(n-2\alpha)(1-q)}} \int_{B(0, |x|)} \frac{d\sigma(y)}{|y|^{q(n-2\alpha)}} \\ & \leq c c_1^{(n-2\alpha)q} \frac{1}{|x|^{(n-2\alpha)(1-q)}} \int_{B(0, |x|)} \frac{d\sigma(y)}{|y|^{q(n-2\alpha)}} \\ & + c \int_{|y| \geq c_1 |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} + c \int_{|y| \geq |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}}. \end{aligned}$$

If $c c_1^{(n-2\alpha)q} \leq \frac{1}{2}$ and $c_1 < 1$, then

$$\frac{1}{|x|^{(n-2\alpha)(1-q)}} \int_{B(0,|x|)} \frac{d\sigma(y)}{|y|^{q(n-2\alpha)}} \leq 4c \int_{|y| \geq c_1 |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}}.$$

This implies that

$$\frac{1}{|x|^{(n-2\alpha)(1-q)}} \int_{B(0,c_1|x|)} \frac{d\sigma(y)}{|y|^{q(n-2\alpha)}} \leq 4c \int_{|y| \geq c_1 |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}}.$$

Letting $\tilde{x} = c_1 x$, we obtain

$$\frac{1}{|\tilde{x}|^{(n-2\alpha)(1-q)}} \int_{B(0,|\tilde{x}|)} \frac{d\sigma(y)}{|y|^{q(n-2\alpha)}} \leq \frac{4c}{c_1^{(n-2\alpha)(1-q)}} \int_{|y| \geq |\tilde{x}|} \frac{d\sigma(y)}{|y|^{n-2\alpha}}.$$

Therefore, (5.4) holds, which completes the proof of Proposition 5.2. \square

In conclusion, we give a counter example showing that there exist radially symmetric σ for which (1.16) holds, and so (5.1) has a nontrivial solution, but condition (1.14), and consequently Brezis–Kamin type estimates (1.12) fail. Let

$$(5.6) \quad \sigma(y) = \begin{cases} \frac{1}{|y|^s \log^\beta \frac{1}{|y|}}, & \text{if } |y| < 1/2, \\ 0, & \text{if } |y| \geq 1/2, \end{cases}$$

where $s = (1-q)n + 2\alpha q$ and $\beta > 1$. Then $2\alpha < s < n$, since $0 < q < 1$ and $0 < 2\alpha < n$.

Clearly, $\int_{|y| \geq 1} \frac{d\sigma(y)}{|y|^{n-2\alpha}} = 0$, and

$$\int_{|y| < 1} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}} = \int_{|y| < \frac{1}{2}} \frac{dy}{|y|^n \log^\beta \frac{1}{|y|}} < \infty.$$

Hence, (1.16) holds, and so (5.1) has a solution by Theorem 5.1.

On the other hand, for $|x|$ small,

$$\int_{B(0,|x|)} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}} = \int_{|y| < |x|} \frac{dy}{|y|^n \log^\beta \frac{1}{|y|}} \approx \log^{1-\beta} \frac{1}{|x|},$$

and

$$\int_{|y| \geq |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \approx \frac{1}{|x|^{(n-2\alpha)(1-q)} \log^\beta \frac{1}{|x|}}.$$

Therefore,

$$\lim_{|x| \rightarrow 0} \frac{\frac{1}{|x|^{(n-2\alpha)(1-q)}} \int_{B(0,|x|)} \frac{d\sigma(y)}{|y|^{q(n-2\alpha)}}}{\int_{|y| \geq |x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}}} = \lim_{|x| \rightarrow 0} \log \frac{1}{|x|} = +\infty.$$

It follows from Proposition 5.2 that (5.3) fails. Then, as shown above, (1.14) fails as well. Hence, by Theorem 3.3 with $p = 2$, the upper estimate in (1.12) is no longer true for any nontrivial solution u of (5.1) in this case. We recall that the lower estimate in (1.12) is always true for any nontrivial supersolution u (see [CV14b]).

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